then the spectral radius of B is

$$\rho(B) = \frac{1}{2}\left[\cos\left(\frac{\pi}{m}\right) + \cos\left(\frac{\pi}{n}\right)\right]$$

The value of ω to be used is, consequently,

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(B)]^2}} = \frac{4}{2 + \sqrt{4 - \left[\cos\left(\frac{\pi}{m}\right) + \cos\left(\frac{\pi}{n}\right)\right]^2}}$$

Solving Parabolic Partial Differential Equations

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < l, \quad t > 0$$

$$u(0,t) = u(l,t) = 0, \quad t > 0, \quad \text{and} \quad u(x,0) = f(x), \quad 0 \le x \le l$$

Forward Difference Method

$$\frac{\partial u}{\partial t}\left(x_i, t^n\right) = \frac{u_i^{n+1} - u_i^n}{\Delta t} \qquad \frac{\partial^2 u}{\partial x^2}\left(x_i, t^n\right) = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\left(\Delta x\right)^2}$$

Dividing the domain into m parts, the discretized form of the heat (diffusion) equation for internal nodes is,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\left(\Delta x\right)^2} \qquad i = 1, 2, \ldots, m-1$$

In the above equation n refers to the current time. Rewriting the equation, we have,

$$u_i^{n+1} = \left(1 - \frac{2\alpha^2 \Delta t}{\left(\Delta x\right)^2}\right) u_i^n + \frac{\alpha^2 \Delta t}{\left(\Delta x\right)^2}\left(u_{i+1}^n + u_{i-1}^n\right) \qquad i = 1, 2, \ldots, m-1$$

Recalling that,

$$u(0,t) = u(l,t) = 0, \quad t > 0, \quad \text{and} \quad u(x,0) = f(x), \quad 0 \le x \le l$$

The discretized equation can be written in the matrix form as,

$$u^{n+1} = Au^{n}$$

where,

$$u^{n} = \begin{bmatrix} u_1^n & u_2^n & \ldots & u_{m-1}^n \end{bmatrix}$$

and,

$$A = \begin{bmatrix} (1-2\lambda) & \lambda & 0 & \cdots & 0 \\ \lambda & (1-2\lambda) & \lambda & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda \\ 0 & \cdots & 0 & \lambda & (1-2\lambda) \end{bmatrix} \qquad \lambda = \alpha^2 \frac{\Delta t}{(\Delta x)^2}$$

Forward difference method is called , also, **explicit method**. Truncation error associated with this method is of order $O(\Delta t + (\Delta x)^2)$.

Example

Use steps sizes **(a)** $\Delta x = 0.1$ and $\Delta t = 0.0005$ and **(b)** $\Delta x = 0.1$ and $\Delta t = 0.01$ to approximate the solution to the heat equation,

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < 1, \quad 0 \le t,$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 < t$$

and initial conditions

$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 1$$

Compare the results at $t = 0.5$ to the exact solution

$$u(x,t) = e^{-\pi^2 t}\sin(\pi x)$$

Solution

(a) Forward-Difference method gives the results in the third column of the following table. These results are quite accurate.

(b) Forward-Difference method gives the results in the fifth column of the following table. These results are worthless.

Illustration

In the case **(b)** unlike the case **(a)**, numerical errors grow by advancing in time and the method becomes **unstable**. So, the stability of the method must be considered.

x_i	*exact* $u(x_i, 0.5)$	u_i^{1000} $k = 0.0005$	$\lvert u(x_i, 0.5) - u_i^{1000} \rvert$	u_i^{50} $k = 0.01$	$\lvert u(x_i, 0.5) - u_i^{50} \rvert$
0.0	0	0		0	
0.1	0.00222241	0.00228652	6.411×10^{-5}	8.19876×10^{7}	8.199×10^{7}
0.2	0.00422728	0.00434922	1.219×10^{-4}	-1.55719×10^{8}	1.557×10^{8}
0.3	0.00581836	0.00598619	1.678×10^{-4}	2.13833×10^{8}	2.138×10^{8}
0.4	0.00683989	0.00703719	1.973×10^{-4}	-2.50642×10^{8}	2.506×10^{8}
0.5	0.00719188	0.00739934	2.075×10^{-4}	2.62685×10^{8}	2.627×10^{8}
0.6	0.00683989	0.00703719	1.973×10^{-4}	-2.49015×10^{8}	2.490×10^{8}
0.7	0.00581836	0.00598619	1.678×10^{-4}	2.11200×10^{8}	2.112×10^{8}
0.8	0.00422728	0.00434922	1.219×10^{-4}	-1.53086×10^{8}	1.531×10^{8}
0.9	0.00222241	0.00228652	6.511×10^{-5}	8.03604×10^{7}	8.036×10^{7}
1.0	0	0		0	

Matrix Norms

A **matrix norm** on the set of all $n \times n$ matrices , $\|\cdot\|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

(i) $\|A\| \geq 0$;

(ii) $\|A\| = 0$, if and only if A is O, the matrix with all 0 entries;

(iii) $\|\alpha A\| = |\alpha| \|A\|$;

(iv) $\|A + B\| \leq \|A\| + \|B\|$;

(v) $\|AB\| \leq \|A\| \|B\|$.

Theorem

If $|| \cdot ||$ is a vector norm on $\mathbb{R}^n$, then

$$\|A\| = \max_{\mathbf{z} \neq \mathbf{0}} \frac{\|A\mathbf{z}\|}{\|\mathbf{z}\|}$$

is a matrix norm.

Corollary

For any vector $\mathbf{z} \neq \mathbf{0}$, matrix A, and any natural norm $\| \cdot \|$,

$$\|A\mathbf{z}\| \leq \|A\| \cdot \|\mathbf{z}\|$$

Theorem

If A is an $n \times n$ matrix, then $\rho(A) \leq \|A\|$

Proof

suppose λ is an eigenvalue of A with eigenvector $\mathbf{x}$ and $\|\mathbf{x}\| = 1$.

Then $A\mathbf{x} = \lambda\mathbf{x}$ and

$$|\lambda| = |\lambda| \cdot \|\mathbf{x}\| = \|\lambda\mathbf{x}\| = \|A\mathbf{x}\| \leq \|A\| \|\mathbf{x}\| = \|A\|$$

Thus

$$\rho(A) = \max |\lambda| \leq \|A\|$$

Stability Considerations

Suppose that an error $\mathbf{e}^{(0)} = (e_1^{(0)}, e_2^{(0)}, \ldots, e_{m-1}^{(0)})^t$ is made in representing the initial data

$$\mathrm{u}^{(0)} = \left(f(x_1), f(x_2), \ldots, f(x_{m-1})\right)^t$$

An error of $A\mathbf{e}^{(0)}$ propagates in $\mathrm{u}^{(1)}$, because

$$\mathrm{u}^{(1)} = \mathrm{A}\left(\mathrm{u}^{(0)} + \mathrm{e}^{(0)}\right) = \mathrm{Au}^{(0)} + \mathrm{Ae}^{(0)}$$

At the nth time step, the error in $\mathrm{u}^{(n)}$ due to $\mathbf{e}^{(0)}$ is $A^n\mathbf{e}^{(0)}$.

The method is **stable** when these errors do not grow as *n* increases,

i.e.,

$$\left\|A^n\mathbf{e}^{(0)}\right\| \leq \left\|\mathbf{e}^{(0)}\right\|$$

Hence, we must have ,

$$||A^n|| \leq 1$$

This condition according to the previous theorem requires that,

$$\rho(A^n) = (\rho(A))^n \leq 1$$

The Forward-Difference method is therefore stable only if,

$$\rho(A) \leq 1$$

The eigenvalues of A can be shown to be ,

$$\mu_i = 1 - 4\lambda\left(\sin\left(\frac{i\pi}{2m}\right)\right)^2, \quad \text{for each } i = 1, 2, \ldots, m-1$$

So the condition for stability consequently reduces to determining whether

$$\rho(A) = \max_{1\leq i\leq m-1}\left|1 - 4\lambda\left(\sin\left(\frac{i\pi}{2m}\right)\right)^2\right| \leq 1$$

and this simplifies to

$$0 \leq \lambda\left(\sin\left(\frac{i\pi}{2m}\right)\right)^2 \leq \frac{1}{2}, \quad \text{for each } i = 1, 2, \ldots, m-1$$

Stability requires that this inequality condition hold as $h \to 0$, or, equivalently, as $m \to \infty$. The fact that

$$\lim_{m\to\infty}\left[\sin\left(\frac{(m-1)\pi}{2m}\right)\right]^2 = 1$$

means that stability will occur only if $0 \leq \lambda \leq \frac{1}{2}$. So, the relation,

$$\alpha^2 \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

expresses the **stability condition** for forward difference method. So, this method is **conditionally stable**. In fact, Δt and Δx must be chosen in such a way to fulfill the stability condition.

Backward-Difference Method

Using backward difference formula for time derivative, the heat equation can be discretized as,

$$\frac{u_i^n - u_i^{n-1}}{\Delta t} = \alpha^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \qquad i = 1, 2, \ldots, m-1$$

This equation can be rewritten as,

$$(1+2\lambda)u_i^n - \lambda u_{i+1}^n - \lambda u_{i-1}^n = u_i^{n-1} \qquad i = 1, 2, \ldots, m-1$$

where, $\lambda = (\alpha^2 \Delta t)/(\Delta x)^2$

Recalling that,

$$u(0,t) = u(l,t) = 0, \quad t > 0, \quad \text{and} \quad u(x,0) = f(x), \quad 0 \le x \le l$$

The discretized equation can be written in the matrix form as,

$$Au^{n} = u^{n-1}$$

where,

$$u^{n} = \begin{bmatrix} u_1^n & u_2^n & \ldots & u_{m-1}^n \end{bmatrix}$$

and,

$$A = \begin{bmatrix} (1+2\lambda) & -\lambda & 0 & \cdots & 0 \\ -\lambda & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\lambda \\ 0 & \cdots & 0 & -\lambda & (1+2\lambda) \end{bmatrix} \qquad \lambda = \alpha^2 \frac{\Delta t}{(\Delta x)^2}$$

Having u^{n-1}, a linear system of equations must be solved to obtain u^n. The backward difference method is called, also, the **implicit method**. The matrix A is **positive definite**, **strictly diagonally dominant** and **tridiagonal**. So, **Crout factorization** (for rather small systems) or **SOR** method (for large systems) can be used to solve the system.

Stability Considerations

$$Au^n = u^{n-1} \Rightarrow u^n = A^{-1}u^{n-1}$$

So, the backward difference method is stable only if, $\rho\left(A^{-1}\right) \le 1$. For the Backward-Difference method, the eigenvalues are

$$\mu_i = 1 + 4\lambda\left[\sin\left(\frac{i\pi}{2m}\right)\right]^2, \quad \text{for each } i = 1, 2, \ldots, m-1$$

Since $\lambda > 0$, so we have $\mu_i > 1$ for all $i = 1, 2, \ldots, m-1$. Since the eigenvalues of A^{-1} are the reciprocals of those of A,

$$\rho(A^{-1}) < 1$$

So, the backward difference method is **unconditionally stable**.

Example

Use the Backward-Difference method with $\Delta x = 0.1$ and $\Delta t = 0.01$ to approximate the solution to the heat equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < 1, \quad 0 < t$$

subject to the constraints

$$u(0,t) = u(1,t) = 0, \quad 0 < t, \quad u(x,0) = \sin \pi x, \quad 0 \le x \le 1$$

Solution

The following table represents the results for the implicit method. As was observed in the previous example, because of the stability problems the explicit method was not able to produce accurate results.

x_i	u_i^{50}	exact $u(x_i, 0.5)$	$\lvert u_i^{50} - u(x_i, 0.5)\rvert$
0.0	0	0	
0.1	0.00289802	0.00222241	6.756×10^{-4}
0.2	0.00551236	0.00422728	1.285×10^{-3}
0.3	0.00758711	0.00581836	1.769×10^{-3}
0.4	0.00891918	0.00683989	2.079×10^{-3}
0.5	0.00937818	0.00719188	2.186×10^{-3}
0.6	0.00891918	0.00683989	2.079×10^{-3}
0.7	0.00758711	0.00581836	1.769×10^{-3}
0.8	0.00551236	0.00422728	1.285×10^{-3}
0.9	0.00289802	0.00222241	6.756×10^{-4}
1.0	0	0	